Lecture 7 De Rham cohomology theory

1. A little homological algebra

- (co)chain cpx (of vector spaces).

$$(C^\cdot, d^\cdot) = \Big(\underset{k}{\bigoplus} C^k , \{ d_k : C^k \to C^{k+1} \}_k \Big)$$

$\bigoplus_k C^k$: graded vector space (not nec f.d.)

s.t. $d_{k+1} \cdot d_k (: C^k \to C^{k+2}) = 0 \quad \forall k.$

e.g. $\quad C^k = \Omega^k(M) = \{ k\text{-forms on } M \}$

$d_k : C^k \to C^{k+1}$ exterior derivative (and $d_{k+1} \circ d_k = 0$ checked earlier).

In particular, $k<0$ or $k > \dim M$, we have $C^k = 0$.

- Given a cochain cpx $C^\cdot = (C^\cdot, d_\cdot)$, define its cohomology groups

by
$$H^k(C^\bullet;\mathbb{K}) := \frac{\ker(d_k : C^k \to C^{k+1})}{\operatorname{Im}(d_{k-1} : C^{k-1} \to C^k)} \quad \forall k \in \mathbb{Z}$$

ground field for C^k

closed elements

exact elements

(note that $d_k \circ d_{k-1} = 0 \Rightarrow \operatorname{Im}(d_{k-1}) \subseteq \ker(d_k)$).

e.g. The k-th de Rham cohomology group is defined from $\Omega^\bullet = (\Omega^\bullet, d_\bullet)$,

$$H^k_{dR}(M;\mathbb{R}) := \frac{\ker(d_k : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{Im}(d_{k-1} : \Omega^{k-1}(M) \to \Omega^k(M))}$$

Then when $k<0$ and $k > \dim M$, $H^k_{dR}(M;\mathbb{R}) = 0$

Thm (de Rham) For each k, $H^k_{dR}(M;\mathbb{R})$ is finite dim'l over $\mathbb{R}$.

Rmk $\dim_{\mathbb{R}} H^k_{dR}(M;\mathbb{R})$ is called the k-th Betti number of M.
(denoted by $b_k(M;\mathbb{R})$)

$$\chi(M) := \sum_{k=0}^{\dim M} (-1)^k \dim_{\mathbb{R}} H^k_{dR}(M;\mathbb{R}) = \sum_{k=0}^{\dim M} (-1)^k b_k(M;\mathbb{R})$$

is called the Euler characteristic of M.

e.g. $H^0_{dR}(M;\mathbb{R}) = \operatorname{Ker}(d_0 : \Omega^0(M;\mathbb{R}) \to \Omega^1(M;\mathbb{R}))$

$= \{ f : M \to \mathbb{R} \mid df \equiv 0 \}$

= constant fcns on each connected compoments of M

$= \mathbb{R}^{\#\text{ connected component of } M} \quad (= \mathbb{R}^{|\pi_0(M)|})$

If M is connected, then $H^0_{dR}(M;\mathbb{R}) = \mathbb{R}$.

e.g. $$H^1_{dR}(M;\mathbb{R}) = \frac{\operatorname{Ker}(d_1 : \Omega^1(M;\mathbb{R}) \to \Omega^2(M;\mathbb{R}))}{\operatorname{Im}(d_0 : \Omega^0(M;\mathbb{R}) \to \Omega^1(M;\mathbb{R}))}$$

In general, we can't say much. Recall we have proved if a 1-form θ vanishes for all loops in M, i.e. $\int_\gamma \theta = 0 \ \forall \gamma$, then $\exists f : M \to \mathbb{R}$, s.t. $\theta = df$.

(c.f.) $\Rightarrow H^*(|pt|) = 0$ for $* > 0$

$\Rightarrow$ if θ is closed ($d_1\theta = 0$) but $[\theta] \neq 0$, then $\exists \gamma$ loop in M s.t. $\int_\gamma \theta \neq 0$.

Exe If $\pi_1(M, x_0)$ is a finite group (for any basept), then $H^1_{dR}(M; \mathbb{R}) = 0$.

- chain map $f: C^\cdot \to D^\cdot$ is a family of $\mathbb{K}$-linear maps

$$f = \{f^k : C^k \to D^k\}_k \quad \text{s.t.} \quad d_k^D \circ f^k = f^{k+1} \circ d_k^C$$

Observe that a chain map $f: C^\cdot \to D^\cdot$ induces a well-defined map on corresponding cohomology groups

$$f_* = \{ f_*^k : H^k(C^\cdot; \mathbb{K}) \to H^k(D^\cdot; \mathbb{K}) \}$$

explicitly given by $f_*^k([x]) := [f^k(x)]$ ← This is well-defined! (DIY)

e.g. $N \xrightarrow{\varphi} M$ a smooth map, then

↙ pullback

$$f_\varphi = \varphi^* : \Omega^\cdot(M) \to \Omega^\cdot(N)$$

Moreover, $d_k^N \circ \varphi^* = \varphi^* \circ d_k^M$, so $f_\varphi = \{f_\varphi^k\}$ is a chain map.

$\Longrightarrow \quad (f_\varphi)^k_* : H^k_{dR}(M;\mathbb{R}) \longrightarrow H^k_{dR}(N;\mathbb{R})$ well-defined (induced from $\varphi : N \to M$).

(C1)

- If $\varphi = \mathbb{1}_M : M \to M$, then $(f_\varphi)^k_* = \mathbb{1} : H^k_{dR}(M;\mathbb{R}) \circlearrowleft$

(C2)

- For maps $N \xrightarrow{\varphi} M \xrightarrow{\psi} L$, b/c $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

 we have $(f_{\psi\circ\varphi})_* = (f_\varphi)_* \circ (f_\psi)_*$.

- A short exact seq of cochain cpxes:

$$0 \longrightarrow C^\bullet \xrightarrow{f^\bullet} D^\bullet \xrightarrow{g^\bullet} E^\bullet \longrightarrow 0 \qquad (*)$$

means $\forall k$, $\ker(g^k) = \operatorname{im}(f^k)$.

Then $(*)$ induces a long exact seq.

$$H^k(C^\bullet;\mathbb{K}) \xrightarrow{f^k} H^k(D^\bullet;\mathbb{K}) \xrightarrow{g^k} H^k(E^\bullet;\mathbb{K}) \xrightarrow{\delta^k} H^{k+1}(C^\bullet;\mathbb{K}) \longrightarrow \cdots \qquad \text{(Exe)}$$

Here, δ^k is called the k-th connecting map. Let us describe δ^k by "diagram chasing".

$$\begin{tikzcd}
0 \arrow[r] & C^k \arrow[r, "f^k"] & D^k \arrow[r, "g^k"] & E^k \arrow[r] & 0 \\
 & & \exists d \arrow[r, "g^k"] \arrow[d, "d^D"] & e \arrow[r, squiggly] \arrow[d, dashed, "d^E"] & {[e] \in H^k(E; \mathbb{K})} \quad (\text{so } d^E(e)=0) \\
 & \exists c \arrow[r, "f^{k+1}"] & d^D(d) \arrow[r, dashed, "g^{k+1}"] & 0 & \\
 & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & \\
0 \arrow[r] & C^{k+1} \arrow[r, "f^{k+1}"'] & D^{k+1} \arrow[r, "g^{k+1}"'] & E^{k+1} \arrow[r] & 0
\end{tikzcd}$$

Moreover, $d^C(c) = 0$ (b/c f^{k+1} is injective).

So $\delta^k([e]) = [c]$ (well-defined!)

e.g. For a submfd $A \subset M$, define (due to <u>Godbillon</u>)

$$\Omega^k(M, A) = \{ k\text{-form on } M \text{ that vanishes on } A \}.$$

Then we have a short exact sequence:

$$\underset{(A,\emptyset)}{\overset{\shortparallel}{A}} \overset{i}{\hookrightarrow} \underset{(M,\emptyset)}{\overset{\shortparallel}{M}} \xrightarrow{j} (M,A) \qquad \big(\text{where } \underset{\text{inclusion of pair}}{j: M (= (M,\emptyset)) \to (M,A)}\big).$$

Then

$$0 \to \Omega^{\cdot}(M,A) \xrightarrow{f \,\swarrow \text{induced by } j} \Omega^{\cdot}(M) \xrightarrow{g \,\swarrow \text{induced by } i} \Omega^{\cdot}(A) \to 0$$

$$\overset{(CA)}{\Longrightarrow} \quad H^k(M,A;\mathbb{K}) \longrightarrow H^k(M;\mathbb{K}) \longrightarrow H^k(A;\mathbb{K}) \xrightarrow{\exists \delta^k} H^{k+1}(M,A;\mathbb{K}) \to \cdots$$

By discussion above, $\delta^k([\theta]) = [\alpha]$ where $\theta \in \Omega^k(A)$ and $\alpha \in \Omega^{k+1}(M,A)$.

e.g. If $M = M_1 \cup M_2$, then $M_1 \overset{i}{\hookrightarrow} M \xrightarrow{q} M_2$. induces

$$0 \to \Omega^{\cdot}(M_2) \to \Omega^{\cdot}(M) \to \Omega^{\cdot}(M_1) \to 0 \quad \text{s.e.s.}$$

$$\Longrightarrow \quad H^k(M_1;\mathbb{K}) \to H^k(M;\mathbb{K}) \to H^k(M_2;\mathbb{K}) \xrightarrow{\delta^k} H^{k+1}(M_1;\mathbb{K}) \to \cdots$$

One can verify that $\delta^k \equiv 0$, so

$$H^k(M;\mathbb{K}) = H^k(M_1;\mathbb{K}) \oplus H^k(M_2;\mathbb{K}) \quad \forall k.$$

Bott-Tu's approach to relative deRham coh group.

$$\Omega^k(M,A) := \Omega^k(M) \oplus \Omega^{k-1}(A) \qquad A \overset{i}{\hookrightarrow} M.$$

define $d_k : \Omega^k(M,A) \longrightarrow \Omega^{k+1}(M,A)$

$$(\omega, \theta) \longrightarrow (d\omega,\ i^*\omega - d\theta)$$

Then

$$(\omega,\theta) \xrightarrow{d_k} (d\omega,\ i^*\omega - d\theta) \xrightarrow{d_{k+1}} (0,\ i^*(d\omega) - d(i^*\omega - d\theta)) = (0,0) \checkmark$$

We have a short exact seq.

redefine $\Omega^\bullet(A)[-1] = \Omega^{\bullet-1}(A)$

$$0 \to \Omega^\bullet(A)[-1] \to \Omega^\bullet(M,A) \longrightarrow \Omega^\bullet(M) \to 0$$

$\Longrightarrow$ induce a long exact sequence

$$\begin{aligned}
&\underset{H^{k-1}(A;\mathbb{R})}{\overset{\shortparallel}{H^k(A;\mathbb{R})[-1]}} \longrightarrow H^k(M,A;\mathbb{R}) \longrightarrow H^k(M;\mathbb{R}) \\
\longrightarrow\ &\underset{H^k(A;\mathbb{R})}{\overset{\shortparallel}{H^{k+1}(A;\mathbb{R})[-1]}} \longrightarrow H^{k+1}(M,A;\mathbb{R}) \longrightarrow \cdots
\end{aligned}$$

connecting morphism.

FACT: These two models give the same long exact seq.

In the second model,

$$H^k(A;\mathbb{R}) \xrightarrow{\delta^k} H^{k+1}(M,A;\mathbb{R})$$

δ^k ← connecting morphism from the first model.

is simply induced by inclusion $\Omega^{k-1}(A) \longrightarrow \Omega^k(M,A)$.

Therefore, given two pairs (N,B) and (M,A) and morphism

$$\varphi: N \to M \text{ with } \varphi|_B (=\psi): B \to A$$

we have the following commutative diagram

(C3)

$$\begin{tikzcd} H^k(B;\mathbb{K}) \arrow[r, "\delta^k_{(N,B)}"] & H^{k+1}(N,B;\mathbb{K}) \\ H^k(A;\mathbb{K}) \arrow[u, "(f_\psi)^k"] \arrow[r, "\delta^k_{(M,A)}"] & H^{k+1}(M,A;\mathbb{K}) \arrow[u, "(f_\varphi)^{k+1}"'] \end{tikzcd}$$

b/c we have the following commutative diagram:

$$\begin{tikzcd} \Omega^{k-1}(B) \arrow[r, "i"] & \Omega^k(N,B) \\ \Omega^{k-1}(A) \arrow[u, "\varphi^*"] \arrow[r, "i"'] \arrow[ur, phantom, "\circlearrowleft"] & \Omega^k(M,A) \arrow[u, "\varphi^*"'] \end{tikzcd}$$

- Two cochain map $f, g : C^\cdot \to D^\cdot$ are chain homotopic if $\exists\ h = \{h^k : C^k \to D^{k-1}\}$

s.t.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{k-1} \arrow[r] & C^k \arrow[r, "d_k^C"] \arrow[dl, "h^k"'] \arrow[d, "f-g"] & C^{k+1} \arrow[r] \arrow[dl, "h^{k+1}"] & \cdots \\
\cdots \arrow[r] & D^{k-1} \arrow[r, "d_{k-1}^D"'] & D^k \arrow[r] & D^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

$$f^k - g^k = h^{k+1} \circ d_k^C - d_{k-1}^D h^k$$

At the first sight, it might look complicated, but it in fact has a strong geometric motivation.

<u>Recall</u> $\varphi, \psi : N \to M$ two continuous maps, they are <u>homotopic</u> if $\exists \{\varphi_t : N \to M\}_{t \in [0,1]}$ s.t $\varphi_0 = \varphi$ and $\varphi_1 = \psi$.

One usually formulate in a different way: $\Phi : [0,1] \times N \to M$

$(t, x) \to \varphi_t(x)$

In particular, the map $\varphi_t : N \to M$ is the composition $\{t\} \times N \xrightarrow{i_t} [0,1] \times N \xrightarrow{\Phi} M$